

DOCUMENT RESUME

ED 090 035

SE 017 625

AUTHOR Shavelson, Richard J.
TITLE Some Methods for Examining Content Structure and Cognitive Structure in Mathematics Instruction.
PUB DATE Apr 74
NOTE 16p.; Paper presented at the annual meeting of the American Educational Research Association (Chicago, Illinois, April 1974)

EDRS PRICE MF-\$0.75 HC-\$1.50 PLUS POSTAGE
DESCRIPTORS *Cognitive Measurement; *Cognitive Processes; Evaluation Methods; *Learning; *Mathematics Education; *Measurement Techniques; Psychometrics; *Structural Analysis

ABSTRACT

Reform in science and mathematics has moved from rote learning of facts and computation skills toward the learning of a structure of a subject matter. At present there is little empirical evidence to support any contentions that there is a match between the subject-structure taught and the cognition in subjects' memories resulting from the instruction. This paper presents methods for examining subject-matter structure in prose material and methods for examining representation of a subject-matter structure in subjects' memories. Data bearing on the validity of structural interpretations of these measures are examined. Such measures provide methods which might be used to evaluate the extent to which a mathematics curriculum communicates the structure it has been developed to communicate. (JP)

SOME METHODS FOR EXAMINING
CONTENT STRUCTURE AND COGNITIVE STRUCTURE
IN MATHEMATICS INSTRUCTION¹

Richard J. Shavelson

University of California at Los Angeles

U.S. DEPARTMENT OF HEALTH,
EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION
THIS DOCUMENT HAS BEEN REPRO-
DUCED EXACTLY AS RECEIVED FROM
THE PERSON OR ORGANIZATION ORIGIN-
ATING IT. POINTS OF VIEW OR OPINIONS
STATED DO NOT NECESSARILY REPRESENT
OFFICIAL NATIONAL INSTITUTE OF
EDUCATION POSITION OR POLICY

Reform in science and mathematics curricula has moved from rote learning of facts and computation skills toward the learning of a structure of a subject matter. The reasons, commonly cited for this shift, are that structural knowledge: (a) is required for a full understanding of the subject matter, (b) leads to solutions of problems not otherwise solvable as Branca (1974) pointed out, (c) leads to an aptitude for learning, and (d) results in intellectual excitement. These reasons for teaching structure are admirable and certainly to be valued, but there is little empirical evidence to support them (for exceptions, see Shavelson, in press).

An initial step in examining the claims for learning structure is to represent, in a clear and objective way, the structure in the to-be-learned subject matter. But a problem arises immediately since a subject-matter structure can be represented in a variety of ways. One approach is to arrange a subject-matter structure hierarchically; the exact arrangement depends upon the psychological model underlying the analysis (cf. Ausubel, 1963; Bruner, 1966; Gagne, 1962). A second approach is content analysis (Berelson, 1954). The universe of content is defined,

¹ Paper presented at the Annual Meetings of the American Educational Research Association, Chicago, April, 1974.

a category system is developed to partition this universe, a unit of analysis is selected, and a measurement is taken.

A third approach is to use a graph structure to represent concepts (broadly defined) and their interrelations in a subject matter (Crothers, 1972; Frederiksen, 1972; Gee, 1973; Shavelson, 1970, 1972). The points or nodes on the graph represent concepts, and the lines represent interrelations between concepts as specified by the syntactic and/or semantic characteristics of (say) the prose materials used to communicate the subject-matter structure.

Schwab (1964) has aptly criticized the hierarchical approach of mapping structure with an underlying psychological model as changing the subject-matter structure itself. Content analysis is too restrictive an approach since it focuses more on, say, words than on their interrelations. Any notion of structure rests at least as much on these interrelations as on the words themselves. The third approach, representing a subject-matter structure as the interrelations among concepts in the subject matter, seems to correspond most closely with what curriculum specialists consider to be subject-matter structure.

Once a method for representing subject-matter structure has been selected, a second problem arises; that of examining the representation of subject-matter structure in the memories of students. Measurement techniques should produce data which are consistent with a specified definition of structure and which can be compared with data representing subject-matter structure. Three such methods are word association, card sorting, and graph building.

The purposes of this paper are to: (a) present a method for examining subject-matter structure in prose material, (b) present methods for

for examining representations of a subject-matter structure in subjects' memories, and (c) present data bearing on the validity of structure interpretations of data obtained from these methods.

Structure: Some Definitions

Structure refers to an assemblage of identifiable elements and the relationships between those elements. Structure may be objective and real or subjective and internal (Shavelson, 1972). The structure in instructional materials is referred to as content structure: the web of concepts (words, symbols) and their interrelations in a body of instructional material. This definition is similar to Schwab's (1962); a discipline is defined as a bundle of facts and the interrelations between those facts. Finally, structure in the student's memory is referred to as cognitive structure: a hypothetical construct referring to the organization (relationships) of concepts in memory.

Method for Examining Content Structure

Content structure may be examined by means of the theory of directed graphs (or digraph theory).¹

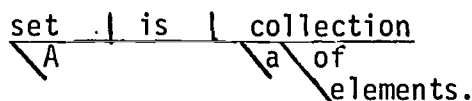
This...theory is concerned with patterns of relationships among pairs of abstract elements. As such, digraph theory makes no reference to the empirical world. Nevertheless, it has potential usefulness to the empirical scientist, for it can serve as a mathematical model of the structural properties of any empirical system consisting of relationships among pairs of elements (Harary, Norman, & Cartwright, 1965, p. 9).

If an appropriate coordination is made so that each entity of an empirical system is identified with a point and each relationship about structural properties of the obtained digraph there are corresponding true statements about structural properties of the empirical system (Harary et al., 1965, p. 22).

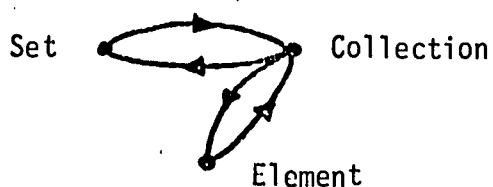
As an example of the application of digraph theory to the examination of subject-matter structure in prose material, consider the subject matter of operational systems. As Branca (1974) pointed out, an operational system is defined as a set together with a binary operation on the set. An operational system may possess any or none of the following properties: associativity, commutativity, identity element, and roundness. When digraphs are used to map structure in instructional material, words denoting concepts in operational systems are represented as points on the digraph and the links between concepts, as specified by syntactical and semantic characteristics of the prose, as directed lines on the graph. The following steps are taken in mapping instructional material onto a digraph:

1. Identify the key concepts in the subject matter. In operational systems, the following concepts were identified by the curriculum developers: Associativity, Binary Operation, Commutativity, Element, Finite/Infinite, Fundamental Properties, Identity Element, Inverses, Ordered Pair, Operational System, Roundness, and Set.

2. If two or more key concepts are found in a sentence, the sentence is diagrammed using the procedure due to Warriner and Griffiths (1957). For example, the sentence, "A set is a collection of elements," was diagrammed as:



3. Using conversion rules for transforming a sentence diagram into a digraph (Shavelson, 1970; Shavelson & Geeslin, 1973), a digraph is obtained for each sentence. In the example, the following digraph was obtained:



The symmetric relation between Set and Collection is specified by the rule for linking verbs: "a linking verb does not specify action and is to be digraphed as a symmetric relation between two points." The symmetric relation between Collection and Element is specified by the rule for prepositions: "if the preposition does not specify direction, the relation is digraphed symmetrically."

4. Each sentence digraph, then, is converted into a distance matrix for all digraphs combined. The following procedure is used. The distance between two points on a digraph is the number of lines in the shortest path connecting the two points. In the digraph above, the distance between Set and Connection is 1, and the distance between Set and Element is 2. Those sentence digraphs were combined into one digraph representing content structure. From this overall digraph a distance matrix was formed using an algorithm given by Harary et al. (1965, pp. 135-136).

5. The data in the distance matrix were considered dissimilarities and examined with Johnson's (1967) hierarchical clustering technique.

Methods for Examining Cognitive Structure

Cognitive structure was examined by the methods of word association, card sorting, and graph building. With these methods, the order of response retrieval or the categorization of concepts was assumed to reflect at least a significant part of cognitive structure. For each test, the 12 key concepts were presented to two persons who had developed

the operational systems curriculum used in the digraph analysis (and in other studies).

For the word-association test, the curriculum developers were instructed to "think like mathematicians" and write down as many words as they could think of which are related to each of the 12 key concepts. They were allowed 1 minute per concept.

For the card-sorting test, they were asked to sort the 12 concepts into piles. Similar concepts were to be placed within a pile. No restriction was put on the number of piles that could be formed.

For the graph-building test, they were asked to construct a linear graph using the 12 concepts. They were instructed to select the 2 most similar concepts, connect them with a line, and label the line "1." Following additional rules, they continued the task until all 12 concepts were connected with 11 lines.

With all three methods, proximity measures between the key concepts were formed. These proximities were examined with Johnson's (1967) hierarchical clustering technique.

Perhaps an example of one method of converting data from a measure of cognitive structure to proximities would be helpful. Consider the word-association data in Table 1. They represent a hypothetical subjects'

Insert Table 1 here.

responses to two concepts in operational system. To the word set, he responds set (implicitly or explicitly), collection, element, finite. To the word element, he responds element (implicitly or explicitly), set, binary operation, finite.

A measure of the similarity between the concepts of set and element, for this subject, is the relatedness coefficient (RC) (Garskoff & Houston, 1963) shown in Table 1. In the numerator, $\bar{A} \cdot \bar{B}$ represents the rank order of words under A which are shared in common with B and the rank order of words in B which are shared in common with A. In Table 1, the common words are set, element, and finite. In the denominator, $A \cdot B$ represents the rank order of words in A multiplied by the rank order of words in B. N represents the number of words in the longer of the two lists, and p represents some fixed number greater than zero which weights various portions of the subject's response lists. We have set $p=1$ which gives equal weight to all portions of the response lists in Table 1. The relatedness coefficient ranges from 0.0 to +1.00; the RC in Table 1 for the concepts of set and element for our hypothetical subject is 0.724.

RESULTS

Content Structure

The largest entry in the distance matrix representing content structure was 5; this value occurred in one cell. These data, then, suggest a tight, formal structure. Since a hierarchical structure was expected, Johnson's (1967) procedure was used to examine the distance data. A meaningful hierarchical structure was retrieved (Figure 1).

 Insert Figure 1 here.

At the apex is one cluster containing the 12 key concepts. From the

apex, two clusters emerged: (a) defining characteristics (BINARY OPERATION, ELEMENT, FINITE/INFINITE, ORDERED PAIR, SET), and (b) fundamental properties (ASSOCIATIVITY, COMMUTATIVITY, FUNDAMENTAL PROPERTIES, IDENTITY ELEMENT, INVERSES, the concept of OPERATIONAL SYSTEM, ROUNDNESS). From the defining characteristics cluster, (a) sets (FINITE/INFINITE, SET) were distinguished from (b) operations on elements (BINARY OPERATION, ELEMENT, ORDERED PAIR). From the fundamental properties cluster, (a) inverses (not in itself a fundamental property but required for the roundness property) and (b) fundamental properties were distinguished. Lower levels make further distinctions which are consistent with our understanding of the subject matter.

Cognitive Structure

One might view prose materials as a measure of the way in which the authors interrelated key concepts in their memories; i.e., a measure of cognitive structure (Shavelson, 1972, in press). The digraph analysis of content structure, then, might be considered a measure of the authors' cognitive structure. If so, then alternative measures of the authors' cognitive structure should provide representations of cognitive structure similar to the one found with the digraph analysis. To examine this hypothesis, data from the three measures of cognitive structure were compared with the data on content structure in Figure 1.

Figures 2 and 3 present the results of the hierarchical cluster analysis of proximity data obtained from the word association and graph building methods. Since both curriculum developers sorted the 12 concepts into two piles--one pile contained the concept of operational system and the other pile contained all of the other concepts--cluster

analysis was unnecessary for these data.

 Insert Figures 2 & 3 here.

A comparison of the results of the cluster analyses for the digraph, word association, and graph building indicated that all three measurement methods produced similar structural representations of the concepts in operational systems. In all cases, the defining characteristics of Operational Systems (Binary Operation, Element, Finite/Infinite, Ordered Pair, and Set) were distinguished from the fundamental properties of Operational Systems [Associativity, Commutativity, Identity Element, and Roundness (cf. Inverses)]. The one exception, Identity Element, was more closely related to defining characteristics than to fundamental properties by the word-association method. This can readily be explained: (a) Identity Element is a particular type of element; and/or (b) Identity Element provided an associative cue for relating this concept to Element.

Additional distinctions among key concepts in the hierarchy were consistent with our understanding of operational systems and slight differences between hierarchies were interpretable. For example, Inverse is separated from fundamental properties in the content structure hierarchy but not in the cognitive structure hierarchies. Inverse is not, formally, a fundamental property of operational systems and this distinction was carefully maintained in the content structure. Inverse, however, is a necessary condition for the Roundness property; Roundness requires that an inverse exist for all pairs of elements in the operational system. Thus, Inverse and Roundness cluster in data from the

measure of cognitive structure.

The methods presented here for examining content and cognitive structure and the data bearing on the validity of structural interpretations of these measures presented here and elsewhere (Geeslin, 1973; Shavelson, 1972, in press; Shavelson & Geeslin, 1973; Shavelson & Stanton, in preparation) indicate that meaningful representations of structure can be obtained. Such measure provide methods which might be used to evaluate the extent to which a mathematics curriculum communicates the structure it has been developed to communicate. They offer useful tools to the researcher concerned with how to communicate a subject-matter structure to a student effectively and efficiently. And, they may even provide diagnostic tools for adapting instructional methods to individual differences in students' knowledge of mathematics.

References

- Ausubel, D. P. Cognitive structure and the facilitation of meaningful verbal learning. Journal of Teacher Education, 1963, 14, 217-221.
- Berelson, B. Content analysis. In G. Lindzey (Ed.), Handbook of social psychology, Vol. I. Cambridge, Mass.: Addison-Wesley, 1954.
- Branca, N. A. Learning mathematical structures. Paper presented at the Annual Meetings of the American Educational Research Association, Chicago, April, 1974.
- Bruner, J. S. Toward a theory of instruction. New York: Norton, 1966.
- Comprehensive School Mathematics Program. Elements of mathematics: Book O. St. Louis: CEMREL, 1971.
- Crothers, E. J. Memory structure and the recall of discourse. In J. B. Carroll and R. O. Freedle (Eds.), Language comprehension and the acquisition of knowledge. New York: Winston & Sons, 1972.
- Dull, C., Metcalfe, H. S., & Williams, J. E. Modern physics. New York: Holt, 1960.
- Fillenbaum, S. & Rapoport, A. Structures in the subjective lexicon. New York: Academic Press, 1971.
- Frederiksen, C. H. Effects of task induced cognitive operations on comprehension and memory processes. In J. B. Carroll & R. O. Freedle (Eds.), Language comprehension and the acquisition of knowledge. New York: Winston & Sons, 1972..
- Gagne, R. M. The acquisition of knowledge. Psychological Review, 1962, 69, 355-365.
- Garskoff, B. E. & Houston, J. P. Measurement of verbal relatedness: An idiographic approach. Psychological Review, 1963, 70, 277-288.
- Geeslin, W. E. An exploratory analysis of content structure and cognitive structure in the context of formative evaluation of a mathematics curriculum. Unpublished doctoral dissertation, Stanford University, 1973.
- Johnson, S. C. Hierarchical clustering schemes. Psychometrika, 1967, 32, 373-380.
- Schwab, J. J. The concept of the structure of a discipline. Educational Record, 1962, 43, 197-205.
- Schwab, J. J. Problems, topics and issues. In S. Elam (Ed.), Education and the structure of knowledge. Chicago: Rand McNally, 1964.

- Shavelson, R. J. Some aspects of the relationship between content structure and cognitive structure in physics instruction. Unpublished doctoral dissertation, Stanford University, 1970.
- Shavelson, R. J. Some aspects of the correspondence between content structure and cognitive structure in physics instruction. Journal of Educational Psychology, 1972, 63, 225-234.
- Shavelson, R. J. Methods for examining representations of a science subject-matter structure in a student's memory. Journal of Research in Science Teaching, in press.
- Shavelson, R. J. & Geeslin, W. E. A method for examining subject-matter structure in written material. Journal of Structural Learning, 1973, 4, 101-109.
- Shavelson, R. J. & Stanton, G. C. Convergent validity of three measures of cognitive structure, in preparation.
- Warriner, J. E. & Griffith, F. English grammar and composition: Complete course. New York: Harcourt, 1957.

Table 1

Data on a Word Association
Test for a Hypothetical Subject
Responding to Concepts in Operational Systems

Response List

Set	A	rank	Element	B	rank
<u>set</u>		4	<u>element</u>		4
collection		3	set		3
element		2	binary operation		2
finite		1	finite		1

$$\text{Relatedness Coefficient (RC)} = \frac{\bar{A} \cdot \bar{B}}{(A \cdot B) - \left[\frac{n}{p} - (n-1) \right]^2}$$

$$RC = \frac{(4 \ 2 \ 1) \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}}{(4 \ 3 \ 2 \ 1) \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} - [4^1 - 3^1]^2}$$

$$RC = \frac{12 + 8 + 1}{(16 + 9 + 4 + 1) - 1} = \frac{21}{29} = 0.724$$

RJS:SB

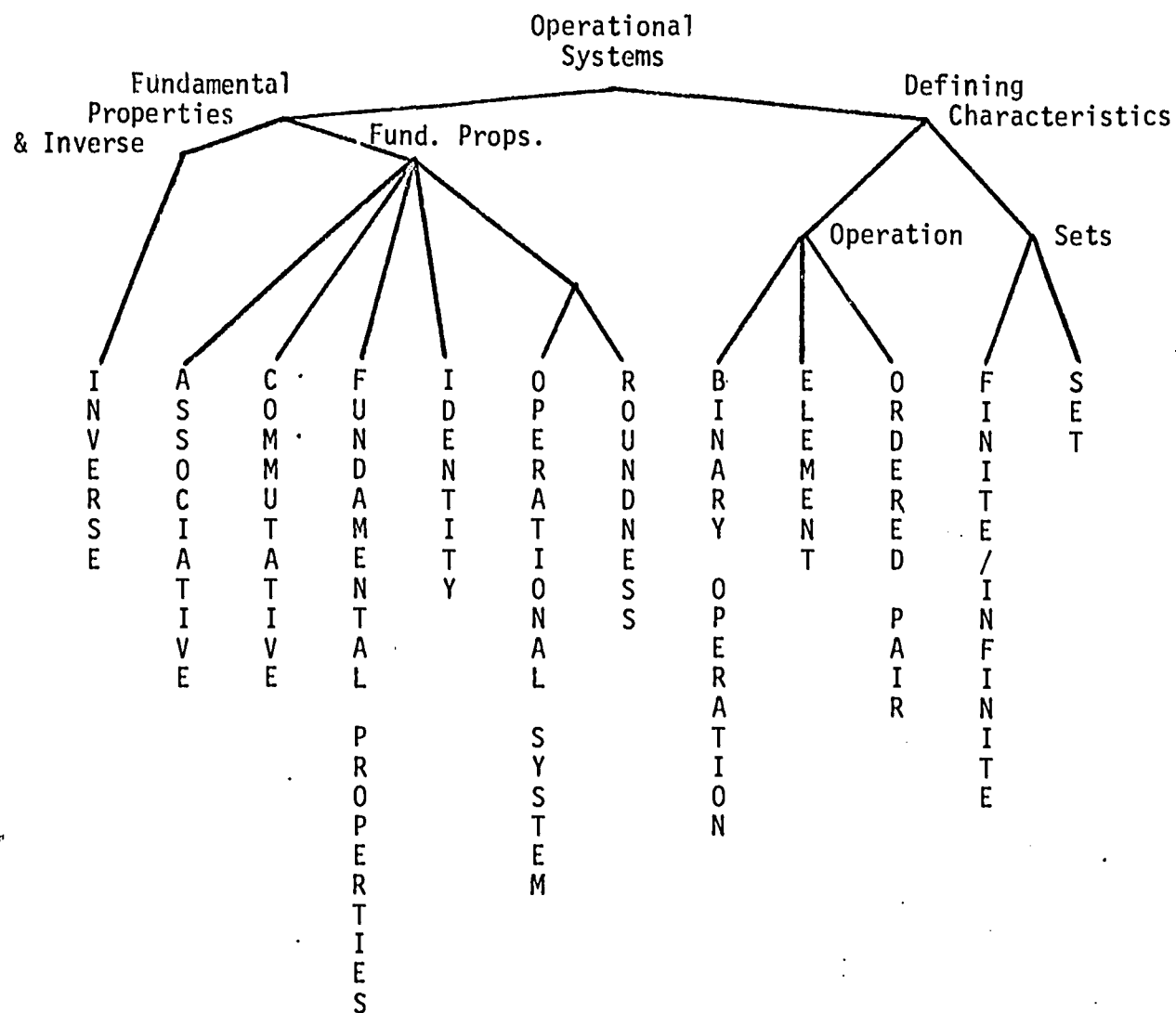


Figure 1.
Digraph Analysis of Content Structure:
Results of the Hierarchical Cluster Analysis of Digraph
Distances Between Key Concepts in Operational Systems

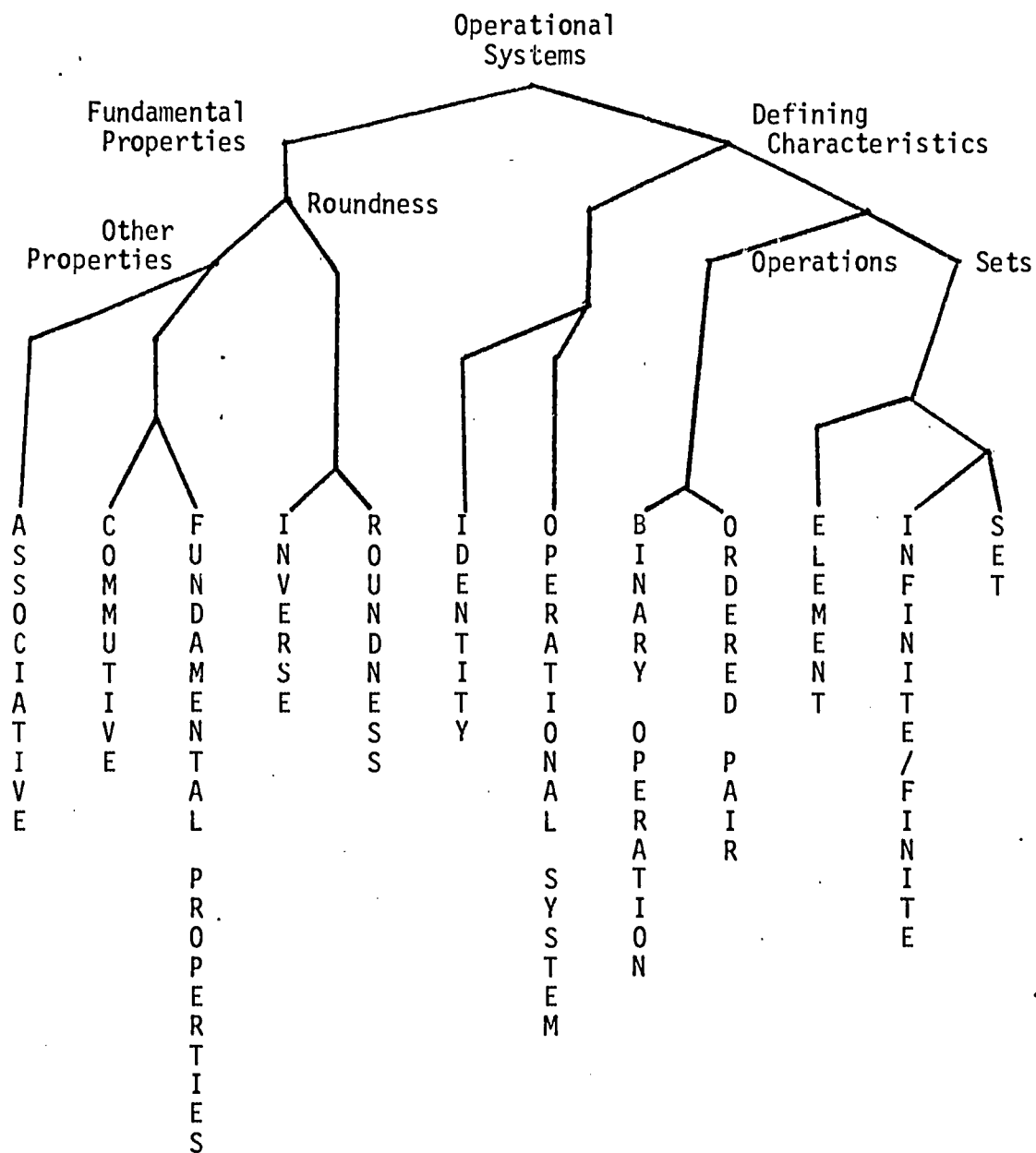


Figure 2.
 Word Association Measure of Cognitive Structure:
 Results of the Hierarchical Cluster Analysis of Proximities
 (Relatedness Coefficients) Between Key Concepts in Operational Systems

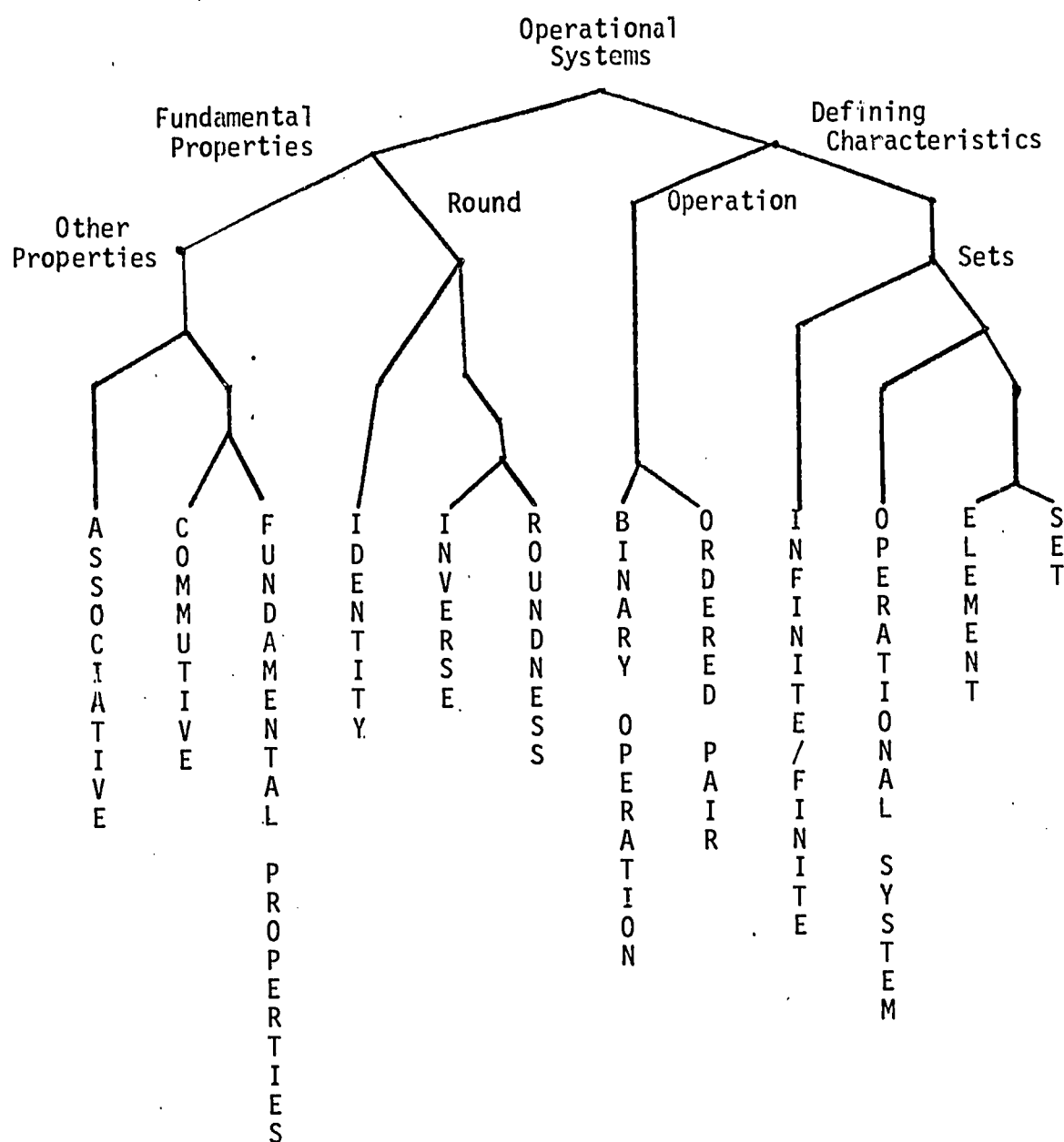


Figure 3.
 Graph Construction Measure of Cognitive Structure:
 Results of the Hierarchical Cluster Analysis of Graph Distances
 Between Key Concepts in Operational Systems